

REFORMULATION OF THE EXTENSION OF THE ν -METRIC FOR H^∞

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ABSTRACT. The classical ν -metric introduced by Vinnicombe in robust control theory for rational plants was extended to classes of nonrational transfer functions in [1]. In [6], an extension of the classical ν -metric was given when the underlying ring of stable transfer functions is the Hardy algebra, H^∞ . However, this particular extension to H^∞ did not directly fit in the abstract framework given in [1]. In this paper we show that the case of H^∞ also fits into the general abstract framework in [1] and that the ν -metric defined in this setting is identical to the extension of the ν -metric defined in [6]. This is done by introducing a particular Banach algebra, which is the inductive limit of certain C^* -algebras.

1. INTRODUCTION

We recall the general stabilization problem in control theory. Suppose that R is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R (thought of as the set of unstable plants). The stabilization problem is then the following: given an unstable plant transfer function $P \in (\mathbb{F}(R))^{p \times m}$, find a stabilizing controller transfer function $C \in (\mathbb{F}(R))^{m \times p}$ such that

$$H(P, C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in R^{(p+m) \times (p+m)}.$$

Robust stabilization goes one step further; in many practical situations one knows that the plant is merely an approximation of reality and therefore one wishes that the controller C not only stabilizes the nominal plant P , but also all plants \tilde{P} , sufficiently close to P . A metric which emerged from the need to define closeness of plants, is the so-called ν -metric, introduced by Vinnicombe in [8], where it was shown that stability is a robust property of the plant with respect to the ν -metric. However, R was essentially taken to be the set of rational functions without poles in the closed unit disk.

In [1] the ν -metric of Vinnicombe was extended in an abstract manner, in order to cover the case when R is a ring of stable transfer functions of possibly infinite-dimensional systems. In particular, the set-up for defining the abstract ν -metric was as follows:

- (A1) R is a commutative integral domain with identity.
- (A2) S is a unital commutative semisimple complex Banach algebra with an involution \cdot^* , such that $R \subset S$.
- (A3) With $\text{inv } S$ denoting the invertible elements of S , there exists a map $\iota : \text{inv } S \rightarrow G$, where (G, \star) is an Abelian group with identity denoted by \circ , and ι satisfies:
 - (I1) $\iota(ab) = \iota(a) \star \iota(b)$ for all $a, b \in \text{inv } S$,
 - (I2) $\iota(a^*) = -\iota(a)$ for all $a \in \text{inv } S$,

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- (I3) ι is locally constant, that is, ι is continuous when G is equipped with the discrete topology.
- (A4) $x \in R \cap \text{inv } S$ is invertible as an element of R if and only if $\iota(x) = \circ$.

In [1], it was shown that the abstract ν -metric defined in the above framework (which is recalled in Definition 4.1 below), is a metric on the class of all stabilizable plants, and moreover, that stabilizability is a robust property of the plant.

In [6], an extension of the ν -metric was given when $R = H^\infty$, the Hardy algebra of bounded and holomorphic functions in the unit disk in \mathbb{C} . However, the ν -metric for H^∞ which was defined there, did not fit in the abstract framework of [1] in a direct manner. Indeed, the metric was defined with respect to a parameter ρ (essentially by using the abstract framework specialized to the disk algebra and looking at an annulus of radii ρ and 1), and then the limit as $\rho \nearrow 1$ was taken to arrive at a definition of an extended ν -metric. (This is recalled in Definition 5.1 below.)

It is a natural question to ask if the extension of the ν -metric for H^∞ given in [6] can be viewed as a special case of the abstract framework in [1] with an appropriate choice of the Banach algebra S and the index function ι . In this paper we shall show that this is indeed possible. Thus our result in this paper gives further support to the abstract framework developed in [1], and progress in the abstract framework of [1] would then also be applicable in particular to our specialization when $R = H^\infty$. We will construct a unital commutative semisimple Banach algebra S and an associated index function $\iota := W$ for which (A1)-(A4) hold. Moreover, we prove that the resulting ν -metric obtained as a result of this specialization of the abstract ν -metric defined in [1] is identical to the extension of the ν -metric defined for H^∞ previously in [6].

The outline of the paper is as follows:

- (1) In Section 2 we introduce some notation.
- (2) In Section 3, when $R = H^\infty$, we construct a certain Banach algebra, $S := \varinjlim C_b(\mathbb{A}_r)$, and an associated index function $\iota := W$ satisfying the assumptions (A1)-(A4).
- (3) In Section 4, we define the ν -metric for H^∞ obtained by specializing the abstract ν -metric of [1] with these choices of $R := H^\infty$, $S := \varinjlim C_b(\mathbb{A}_r)$ and $\iota := W$.
- (4) In Section 5, we prove that the ν -metric obtained for H^∞ in this setup coincides with the ν -metric for H^∞ given in [6].
- (5) Finally, in Section 6, as an illustration of the computability of the proposed ν -metric, we give an example where we calculate the ν -metric when there is uncertainty in the location of the zero of the (nonrational) transfer function.

2. NOTATION

In this section we will fix some notation which will be used throughout the article.

Let \cdot^* denote the involution in the Banach algebra, mentioned in (A2). For $F \in S^{p \times m}$, the notation $F^* \in S^{m \times p}$ denotes the matrix given by $(F^*)_{ij} = (F_{ji})^*$ for $1 \leq i \leq p$ and $1 \leq j \leq m$. Here $(\cdot)_{ij}$ is used to denote the entry in the i th row and j th column of a matrix.

Let $\mathbb{F}(R)$ denote the field of fractions of R . Given a matrix $P \in (\mathbb{F}(R))^{p \times m}$, a factorization $P = ND^{-1}$, where N and D are matrices with entries from R , is called a *right coprime factorization* of P if there exist matrices X, Y with entries from R , such that $XN + YD = I_m$. If, in addition, $N^*N + D^*D = I_m$, then the right coprime factorization is referred to as a *normalized right coprime factorization* of P .

Given a matrix $P \in (\mathbb{F}(R))^{p \times m}$, a factorization $P = \tilde{D}^{-1} \tilde{N}$, where \tilde{D} and \tilde{N} are matrices with entries from R , is called a *left coprime factorization* of P if there exist matrices \tilde{X}, \tilde{Y} with entries from R , such that $\tilde{N} \tilde{X} + \tilde{D} \tilde{Y} = I_p$. If, in addition, $\tilde{N} \tilde{N}^* + \tilde{D} \tilde{D}^* = I_p$, then the left coprime factorization is referred to as a *normalized left coprime factorization* of P .

Let $\mathbb{S}(R, p, m)$ denote the set of all elements $P \in (\mathbb{F}(R))^{p \times m}$ that possess normalized right- and left coprime factorizations. For $P \in \mathbb{S}(R, p, m)$, with factorizations $P = \tilde{D}^{-1} \tilde{N} = ND^{-1}$, G and \tilde{G} are defined by

$$G := \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{and} \quad \tilde{G} := \begin{bmatrix} -\tilde{D} & \tilde{N} \end{bmatrix}.$$

Further, we will define a norm on matrices with entries in S using the Gelfand transform.

Definition 2.1. Let $\mathfrak{M}(S)$ denote the maximal ideal space of the Banach algebra S . For a matrix $M \in S^{p \times m}$, we define

$$\|M\|_{S, \infty} = \max_{\varphi \in \mathfrak{M}(S)} \|\mathbf{M}(\varphi)\|, \quad (2.1)$$

where \mathbf{M} denotes the entry-wise Gelfand transform of M , and $\|\cdot\|$ denotes the induced operator norm from \mathbb{C}^m to \mathbb{C}^p . (For the sake of concreteness, we assume that \mathbb{C}^m and \mathbb{C}^p are both equipped with the usual Euclidean 2-norm.)

The maximum in (2.1) exists since $\mathfrak{M}(S)$ is a compact space when equipped with the Gelfand topology, that is, the weak-* topology induced from $\mathcal{L}(S; \mathbb{C})$, the set of continuous linear functionals from S to \mathbb{C} . Moreover, since S is semisimple, the Gelfand transform,

$$\hat{\cdot} : S \rightarrow \hat{S} \subset C(\mathfrak{M}(S), \mathbb{C}),$$

is an injective algebra homomorphism by the Gelfand-Naimark theorem.

3. VALIDITY OF (A1)-(A4) WITH $R = H^\infty$, $S = \varinjlim C_b(\mathbb{A}_r)$ AND $\iota = W$

In this section we give a Banach algebra S and an index function ι such that the assumptions (A1)-(A4) are satisfied for $R = H^\infty$.

In order to construct S , will use the notion of inductive limits of C^* -algebras. We refer the reader to [2, Section 2.6] and [9, Appendix L] for background on the inductive limit of C^* -algebras.

The *Hardy algebra* H^∞ consists of all bounded and holomorphic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with pointwise operations and the usual supremum norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|, \quad f \in H^\infty.$$

For given $r \in (0, 1)$, let

$$\mathbb{A}_r := \{z \in \mathbb{C} : r < |z| < 1\}$$

denote the open annulus and let $C_b(\mathbb{A}_r)$ be the C^* -algebra of all bounded and continuous functions $f : \mathbb{A}_r \rightarrow \mathbb{C}$, equipped with pointwise operations and the supremum norm: for $f \in C_b(\mathbb{A}_r)$ we define

$$\|f\|_{L^\infty(\mathbb{A}_r)} := \sup_{z \in \mathbb{A}_r} |f(z)|.$$

When \mathbb{A}_r is implicitly understood we will write $\|\cdot\|_{L^\infty}$ instead of $\|\cdot\|_{L^\infty(\mathbb{A}_r)}$. Moreover, for $0 < r \leq R < 1$ we define the map $\pi_r^R : C_b(\mathbb{A}_r) \rightarrow C_b(\mathbb{A}_R)$ by restriction:

$$\pi_r^R(f) = f|_{\mathbb{A}_R}, \quad f \in C_b(\mathbb{A}_r).$$

Consider the family $(C_b(\mathbb{A}_r), \pi_r^R)$ for $0 < r \leq R < 1$. We note that

- (i) π_r^r is the identity map on $C_b(\mathbb{A}_r)$, and
- (ii) $\pi_r^R \circ \pi_\rho^r = \pi_\rho^R$ for all $0 < \rho \leq r \leq R < 1$.

Now consider the $*$ -algebra

$$\prod_{r \in (0,1)} C_b(\mathbb{A}_r),$$

and denote by \mathcal{A} its $*$ -subalgebra consisting of all elements $f = (f_r) = (f_r)_{r \in (0,1)}$ such that there is an index r_0 with $\pi_r^R(f_r) = f_R$ for all $0 < r_0 \leq r \leq R < 1$. Since every π_r^R is norm decreasing, the net $(\|f_r\|_{L^\infty(\mathbb{A}_r)})$ is convergent and we define

$$\|f\| := \lim_{r \rightarrow 1} \|f_r\|_{L^\infty(\mathbb{A}_r)}.$$

Clearly this defines a seminorm on \mathcal{A} that satisfies the C^* -norm identity, that is,

$$\|f^* f\| = \|f\|^2,$$

where $*$ is the involution, that is, complex conjugation, see (3.2) below. Now, if N is the kernel of $\|\cdot\|$, then the quotient \mathcal{A}/N is a C^* -algebra (and we denote the norm again by $\|\cdot\|$). This algebra is the so-called *direct* or *inductive limit* of $(C_b(\mathbb{A}_r), \pi_r^R)$ and we denote it by

$$\varinjlim C_b(\mathbb{A}_r).$$

To every element $f \in C_b(\mathbb{A}_{r_0})$, we associate a sequence $f_1 = (f_r)$ in \mathcal{A} , where

$$f_r = \begin{cases} 0 & \text{if } 0 < r < r_0, \\ \pi_{r_0}^r(f) & \text{if } r_0 \leq r < 1. \end{cases} \quad (3.1)$$

We also define a map $\pi_r : C_b(\mathbb{A}_r) \rightarrow \varinjlim C_b(\mathbb{A}_r)$ by

$$\pi_r(f) := [f_1], \quad f \in C_b(\mathbb{A}_r),$$

where $[f_1]$ denotes the equivalence class in $\varinjlim C_b(\mathbb{A}_r)$ which contains f_1 . We will use the fact that the maps π_r are in fact $*$ -homomorphisms. We note that these maps are compatible with the connecting maps π_r^R in the sense that every diagram shown below is commutative.

$$\begin{array}{ccc} C_b(\mathbb{A}_r) & \xrightarrow{\pi_r^R} & C_b(\mathbb{A}_R) \\ & \searrow \pi_r & \downarrow \pi_R \\ & & \varinjlim C_b(\mathbb{A}_r) \end{array}$$

3.1. Verification of assumption (A2). We note that $\varinjlim C_b(\mathbb{A}_r)$ is a complex commutative Banach algebra with involution, see for instance [2, Section 2.6]. The multiplicative identity arises from the constant function $f \equiv 1$ in $C_b(\mathbb{A}_0)$, that is, $\pi_0(f)$. Moreover, we can define an involution in $C_b(\mathbb{A}_r)$ by setting

$$(f^*)(z) := \overline{f(z)}, \quad z \in \mathbb{A}_r, \quad (3.2)$$

and this implicitly defines an involution of elements in $\varinjlim C_b(\mathbb{A}_r)$.

It remains to prove that $\varinjlim C_b(\mathbb{A}_r)$ is semisimple and that $H^\infty \subset \varinjlim C_b(\mathbb{A}_r)$. Recall that a Banach algebra is *semisimple* if its radical ideal (that is, the intersection of all its maximal ideals) is zero. As $\varinjlim C_b(\mathbb{A}_r)$ is in fact a C^* -algebra, by the Gelfand-Naimark theorem we know that the Gelfand transform is an isometric isomorphism of $\varinjlim C_b(\mathbb{A}_r)$ onto $C(\Delta)$, where Δ is maximal ideal space of $\varinjlim C_b(\mathbb{A}_r)$. Since the maximal ideal space of $C(\Delta)$ comprises just point evaluations at $x \in \Delta$, $C(\Delta)$ is semisimple, and it follows that $\varinjlim C_b(\mathbb{A}_r)$ is semisimple too.

Finally, note that there is a natural embedding of H^∞ into $\varinjlim C_b(\mathbb{A}_r)$, namely

$$f \mapsto \pi_0(f) : H^\infty \longrightarrow \varinjlim C_b(\mathbb{A}_r). \quad (3.3)$$

This is an injective map since π_0 is linear and if $\pi_0(f) = [(0)]$ for $f \in H^\infty$, then

$$\lim_{r \rightarrow 1} \left(\sup_{z \in \mathbb{A}_r} |f(z)| \right) = 0,$$

and so, in particular, the radial limit

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = 0$$

for all $\theta \in [0, 2\pi)$. By the uniqueness of the boundary function for H^∞ functions (see for example [4, Theorem 17.18]), this implies that $f = 0$ in H^∞ . Thus H^∞ can be considered to be a subset of $\varinjlim C_b(\mathbb{A}_r)$ (via the injective restriction of map π_0 to H^∞).

3.2. Verification of assumption (A3). We now construct an index function

$$\iota : \text{inv} \left(\varinjlim C_b(\mathbb{A}_r) \right) \rightarrow G,$$

with a certain choice of an Abelian group (G, \star) satisfying (I1)-(I3). We will take the Abelian group (G, \star) to be the additive group $(\mathbb{Z}, +)$ of integers and we will define ι in terms of winding numbers.

Let $C(\mathbb{T})$ denote the Banach algebra of complex valued continuous functions on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. If $f \in \text{inv} C(\mathbb{T})$, we denote by $w(f) \in \mathbb{Z}$ its winding number, see for instance [7, p. 57]. For $f \in \text{inv} (C_b(\mathbb{A}_\rho))$ and for $0 < \rho < r < 1$ we define the map $f^r : \mathbb{T} \rightarrow \mathbb{C}$ by

$$f^r(\zeta) = f(r\zeta), \quad \zeta \in \mathbb{T}.$$

If $f \in \text{inv} (C_b(\mathbb{A}_\rho))$, then $f^r \in \text{inv} (C(\mathbb{T}))$, and this implies that f^r has a well defined integral winding number $w(f^r) \in \mathbb{Z}$ with respect to 0. In [6, Proposition 3] it is proved that for $f \in \text{inv}(C_b(\mathbb{A}_\rho))$ and $0 < \rho < r < r' < 1$,

$$w(f^r) = w(f^{r'}), \quad (3.4)$$

by the local constancy of the winding number.

Let $[(f_r)] \in \text{inv}(\varinjlim C_b(\mathbb{A}_r))$. Then there exists $[(g_r)] \in \text{inv}(\varinjlim C_b(\mathbb{A}_r))$ such that

$$[(f_r)][(g_r)] = [(1)].$$

Thus there exist $(f_r^1) \in [(f_r)]$, $(g_r^1) \in [(g_r)]$ and r_0 such that for $r \in (r_0, 1)$,

$$f_r^1 g_r^1 = 1$$

pointwise. In particular, the image $f_r^1(\mathbb{A}_r)$, of f_r^1 is a set in \mathbb{C} that is bounded away from zero, that is, there exists a $\delta > 0$ such that $f_r^1(\mathbb{A}_r) \cap \{z \in \mathbb{C} : |z| < \delta\} = \emptyset$. If $(f_r^2) \in [(f_r)]$ is another sequence, then there exists $\tilde{r}_0 \in (0, 1)$ such that $\sup_{z \in \mathbb{A}_r} |f_r^1(z) - f_r^2(z)| < \delta/2$ for all $r \in (\tilde{r}_0, 1)$. Therefore, if we let r be such that $\max\{r_0, \tilde{r}_0\} < r < 1$, we can look at the restrictions

$$f_r^1|_{\mathbb{T}_\rho}, f_r^2|_{\mathbb{T}_\rho},$$

and these will have the same winding number, since the graph of $f_r^1|_{\mathbb{T}_\rho}$ is at least at a distance δ from the origin, while the distance from $f_r^2|_{\mathbb{T}_\rho}$ to $f_r^1|_{\mathbb{T}_\rho}$ is smaller than $\delta/2$. Thus $f_r^2|_{\mathbb{T}_\rho}$ must wind around the origin the same number of times as $f_r^1|_{\mathbb{T}_\rho}$ and their winding numbers coincide, see [7, Proposition 4.12]. With this in mind, we define the map $W : \text{inv}(\varinjlim C_b(\mathbb{A}_r)) \rightarrow \mathbb{Z}$ by

$$W(f) = \lim_{r \rightarrow 1} w(f_r|_{\mathbb{T}_\rho}), \text{ for } f = [(f_r)] \in \text{inv}(\varinjlim C_b(\mathbb{A}_r)), \rho \in (r, 1). \quad (3.5)$$

Since two sequences in the same equivalence class will have the same winding number eventually as $r \nearrow 1$, it is enough to consider only one of the sequences in the equivalence class in (3.5). We take $\iota = W$. By (3.4), (3.5) and the definition of winding numbers, it follows that (I1) and (I2) hold. Finally, analogous to the proof of [6, Proposition 6], it can be verified that assumption (I3) also holds.

3.3. Verification of assumption (A4). In light of (3.3), we can view H^∞ as a subset of $\varinjlim C_b(\mathbb{A}_r)$.

Let $f \in H^\infty \cap \text{inv}(\varinjlim C_b(\mathbb{A}_r))$. First, assume that f is invertible in H^∞ and let $g \in H^\infty$ be its inverse. For each $r \in (0, 1)$ we can define $f^r(z) := f(rz) \in A(\mathbb{D})$, and since f is invertible in H^∞ , f^r is invertible in $A(\mathbb{D})$ and on $C(\mathbb{T})$. By the Nyquist criterion [1, Lemma 5.2] for $A(\mathbb{D})$ this implies that $w(f^r) = 0$. Because of the homotopic invariance of winding numbers, $w(f^r) = w(f|_{\mathbb{T}_r})$, and this implies that

$$W(f) = \lim_{r \rightarrow 1} w(f|_{\mathbb{T}_r}) = 0.$$

Next, assume that $f \in H^\infty \cap \text{inv}(\varinjlim C_b(\mathbb{A}_r))$ and that $W(f) = 0$. Let $F = \pi_0(f)$ and let $G \in \varinjlim C_b(\mathbb{A}_r)$ be the inverse of F . Again, for $r \in (0, 1)$ we define $f^r(z) := f(rz) \in A(\mathbb{D})$. Since $\pi_0(f) \in \text{inv}(\varinjlim C_b(\mathbb{A}_r))$, we have that $f^r \in \text{inv}(C(\mathbb{T}))$. We know that

$$W(f) = \lim_{r \rightarrow 1} w(f_r|_{\mathbb{T}_\rho}) = 0.$$

Using the fact that the winding number is integer valued, and using the local constancy of winding numbers, it follows that $w(f^r|_{\mathbb{T}}) = 0$ for r close enough to 1. Moreover, the Nyquist criterion referred to above implies that f^r is invertible in $A(\mathbb{D})$. In particular, this means that $f(rz) \neq 0$ for all $z \in \mathbb{D}$. Since this is the case for all r large enough, $f(z) \neq 0$ for all $z \in \mathbb{D}$. This implies that f has a pointwise inverse, say g , and this g is holomorphic. What remains to be proved is that g is bounded. To this end, we consider $(f_r) \in F$ as defined in (3.1), and its inverse $(g_r) \in G$, and we note that there exists $\rho \in (0, 1)$ such that for all $r \in (\rho, 1)$,

$$f_r(z)g_r(z) = f(z)g(z) = 1, \quad z \in \mathbb{A}_r.$$

The maximum modulus principle then gives us that

$$\sup_{\mathbb{D}} |g(z)| = \sup_{\mathbb{A}_r} |g(z)| \leq \|g_r\|_{L^\infty(\mathbb{A}_r)} < +\infty.$$

That is, g is bounded and hence $g \in H^\infty$.

Summarizing, in this section we have checked that with

$$R := H^\infty, \quad S := \varinjlim C_b(\mathbb{A}_r), \quad \text{and} \quad \iota := W,$$

the assumptions (A1)-(A4) from [1] (which we recalled at the outset) are all satisfied, and so the abstract ν -metric given in [1] is applicable when the ring of stable transfer functions is the Hardy algebra H^∞ . In the next section, we will clarify the explicit form taken by abstract ν -metric in this specialization when $(R, S, \iota) = (H^\infty, \varinjlim C_b(\mathbb{A}_r), W)$.

4. THE ABSTRACT ν -METRIC WHEN $(R, S, \iota) = (H^\infty, \varinjlim C_b(\mathbb{A}_r), W)$

The abstract ν -metric from [1], when applied to our special case

$$(R, S, \iota) = (H^\infty, \varinjlim C_b(\mathbb{A}_r), W)$$

gives the following ν -metric for stabilizable plants over H^∞ .

Definition 4.1. For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, with normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \tilde{D}_2^{-1} \tilde{N}_2, \end{aligned}$$

the ν -metric d_ν is given by

$$d_\nu(P_1, P_2) = \begin{cases} \|\tilde{G}_2 G_1\|_{\varinjlim C_b(\mathbb{A}_r), \infty} & \text{if } \det(G_1^* G_2) \in \text{inv}(\varinjlim C_b(\mathbb{A}_r)) \text{ and } W(\det(G_1^* G_2)) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Although the normal coprime factorization is not unique for a given plant, d_ν is still a well-defined metric on $\mathbb{S}(R, p, m)$; see [1, Theorem 3.1].

The next step is to show that, for $\tilde{G}_2 G_1 \in (H^\infty)^{p \times m}$, the norm

$$\|\tilde{G}_2 G_1\|_{\varinjlim C_b(\mathbb{A}_r), \infty}$$

above can be replaced by the usual H^∞ -norm $\|\tilde{G}_2 G_1\|_\infty$. This will simplify the calculation of the ν -metric, and will also help us to show that the ν -metric defined above and the extension of the ν -metric for H^∞ given in in [6] are the same. We show the following, analogous to the result in [6, Lemma 3.12].

Theorem 4.2. For $F \in (H^\infty)^{p \times m}$, there holds

$$\|[(F_r)]\|_{\varinjlim C_b(\mathbb{A}_r), \infty} = \|F\|_\infty := \sup_{z \in \mathbb{D}} |F(z)|.$$

Proof. Suppose first that $p = m = 1$. Then

$$\begin{aligned} \|F\|_\infty &= \sup_{z \in \mathbb{D}} |F(z)| = \lim_{r \rightarrow 1} \sup_{z \in \mathbb{A}_r} |F(z)| = \|\pi_0(F)\|_{\varinjlim C_b(\mathbb{A}_r)} \\ &= \max_{\varphi \in \mathfrak{M}(\varinjlim C_b(\mathbb{A}_r))} \|\widehat{\pi_0(F)}(\varphi)\| = \|\pi_0(F)\|_{\varinjlim C_b(\mathbb{A}_r), \infty}. \end{aligned}$$

The last equality follows from the Gelfand-Naimark Theorem; see [5, Theorem 11.18]. This proves the theorem for $p = m = 1$.

Lets assume that at least one of p and m are larger than 1. To treat this case, we introduce the notation $\sigma_{\max}(X)$ for $X \in \mathbb{C}^{p \times m}$, denoting the largest singular value of X , that is, the square root of the largest eigenvalue of XX^* (or X^*X). In particular, we note that the map $\sigma_{\max}(\cdot) : \mathbb{C}^{p \times m} \rightarrow [0, \infty)$ is continuous. Let

$$F = [(F_r)] \in \left(\varinjlim C_b(\mathbb{A}_r) \right)^{p \times m}.$$

Then $\sigma_{\max}(\widehat{F}(\cdot))$ is a continuous function on the maximal ideal space $\mathfrak{M}(\varinjlim C_b(\mathbb{A}_r))$, and so again by the Gelfand-Naimark Theorem, there exists an element $\mu_1 \in \varinjlim C_b(\mathbb{A}_r)$ such that

$$\widehat{\mu}_1(\varphi) = \sigma_{\max}(\widehat{F}(\varphi)) \quad \text{for all } \varphi \in \mathfrak{M}(\varinjlim C_b(\mathbb{A}_r)).$$

Define $\mu_2 := [(\sigma_{\max}(F_r(\cdot)))] \in \varinjlim C_b(\mathbb{A}_r)$. For fixed r , we have that $\det((\mu_2)_r)^2 I - F_r^* F_r = 0$ in $C_b(\mathbb{A}_r)$, which implies that $\det(\mu_2^2 I - F^* F) = 0$ in $\varinjlim C_b(\mathbb{A}_r)$. Taking Gelfand transforms, we obtain

$$\det\left((\widehat{\mu}_2(\varphi))^2 I - (\widehat{F}(\varphi))^* (\widehat{F}(\varphi))\right) = 0$$

and so $|\widehat{\mu}_2(\varphi)| \leq \sigma_{\max}(\widehat{F}(\varphi)) = \widehat{\mu}_1(\varphi)$ for all $\varphi \in \mathfrak{M}(\varinjlim C_b(\mathbb{A}_r))$. Thus

$$\|\mu_2\|_{\varinjlim C_b(\mathbb{A}_r)} \leq \|\mu_1\|_{\varinjlim C_b(\mathbb{A}_r)}. \quad (4.1)$$

On the other hand, since

$$\det\left((\widehat{\mu}_1(\varphi))^2 I - (\widehat{F}(\varphi))^* (\widehat{F}(\varphi))\right) = 0 \quad \text{for all } \varphi \in \mathfrak{M}(\varinjlim C_b(\mathbb{A}_r)),$$

it follows that $\det(\mu_1^2 I - F^* F) = 0$ in $\varinjlim C_b(\mathbb{A}_r)$. Hence, for all $\epsilon > 0$, there exists $r_0 \in (0, 1)$ such that for all $r > r_0$, $|(\mu_1)_r(z)| \leq \sigma_{\max}(F_r(z)) + \epsilon = (\mu_2)_r(z) + \epsilon$, for $z \in \mathbb{A}_r$. So

$$\|\mu_1\|_{\varinjlim C_b(\mathbb{A}_r)} \leq \|\mu_2\|_{\varinjlim C_b(\mathbb{A}_r)} + \epsilon. \quad (4.2)$$

As the choice of ϵ was arbitrary, (4.1) and (4.2) imply that $\|\mu_1\|_{\varinjlim C_b(\mathbb{A}_r)} = \|\mu_2\|_{\varinjlim C_b(\mathbb{A}_r)}$. Using this observation, we have that

$$\begin{aligned} \|[(F_r)]\|_{\varinjlim C_b(\mathbb{A}_r), \infty} &= \max_{\varphi \in \mathfrak{M}(\varinjlim C_b(\mathbb{A}_r))} \widehat{\mu}_1(\varphi) \quad (\text{definition}) \\ &= \|\mu_1\|_{\varinjlim C_b(\mathbb{A}_r)} \\ &= \|\mu_2\|_{\varinjlim C_b(\mathbb{A}_r)} \\ &= \lim_{r \rightarrow 1} \|\sigma_{\max}(F_r(\cdot))\|_{L^\infty(\mathbb{A}_r)} \\ &= \lim_{r \rightarrow 1} \sup_{z \in \mathbb{A}_r} \sigma_{\max}(F_r(z)) \\ &= \lim_{r \rightarrow 1} \sup_{z \in \mathbb{A}_r} \|F_r(z)\| \\ &= \|F\|_\infty \quad (\text{since } F \in (H^\infty)^{p \times m}) \end{aligned}$$

This completes the proof. □

Hence abstract ν -metric from [1], when applied to our special case

$$(R, S, \iota) = (H^\infty, \varinjlim C_b(\mathbb{A}_r), W)$$

now takes the following explicit form.

Definition 4.3. For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, with normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \tilde{D}_2^{-1} \tilde{N}_2, \end{aligned} \tag{4.3}$$

the ν -metric d_ν is given by

$$d_\nu(P_1, P_2) = \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if } \det(G_1^* G_2) \in \text{inv}(\varinjlim C_b(\mathbb{A}_r)) \text{ and } W(\det(G_1^* G_2)) = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

5. THE ν -METRIC FOR H^∞ GIVEN BY (4.4) COINCIDES WITH THE ONE GIVEN IN [6]

The aim with this section is to prove that the extension of the ν -metric given in [6] coincides with the ν -metric given by (4.4), which, as we have seen in the previous section, is a specialization of the abstract ν -metric defined in [1] when $(R, S, \iota) = (H^\infty, \varinjlim C_b(\mathbb{A}_r), W)$.

Let us first recall the extension of the ν -metric for H^∞ given in [6]. In order to distinguish it from the metric d_ν given by (4.4), we denote the metric from [6] by \tilde{d}_ν .

Definition 5.1. For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, with normalized left/right coprime factorizations as in (4.3), let

$$\tilde{d}_\nu^\rho(P_1, P_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if } \det(G_1^* G_2) \in \text{inv } C_b(\mathbb{A}_\rho) \text{ and } w(\det(G_1^* G_2)|_{\mathbb{T}_r}) = 0, \ r \in (\rho, 1), \\ 1 & \text{otherwise.} \end{cases}$$

Then, the extended ν -metric for H^∞ is defined by

$$\tilde{d}_\nu(P_1, P_2) := \lim_{\rho \rightarrow 1} \tilde{d}_\nu^\rho(P_1, P_2). \tag{5.1}$$

Theorem 5.2. *On the set $\mathbb{S}(H^\infty, p, m)$ of stabilizable plants, the metric d_ν given by in (4.4), and the metric \tilde{d}_ν given by (5.1), coincide.*

Proof. First, suppose that $d_\nu(P_1, P_2) < 1$. Then we have $\det(G_1^* G_2) \in \text{inv}(\varinjlim C_b(\mathbb{A}_r))$ and $W(\det(G_1^* G_2)) = 0$. Since $\det(G_1^* G_2)$, viewed as an element of $\varinjlim C_b(\mathbb{A}_r)$ via the map π_0 , belongs to $\text{inv}(\varinjlim C_b(\mathbb{A}_r))$, there exists an equivalence class $[(F_r)] \in \text{inv}(\varinjlim C_b(\mathbb{A}_r))$ such that $\det(G_1^* G_2) \cdot [(F_r)] = [(1)]$. In particular, this means that for $r \in (0, 1)$ large enough $\det(G_1^* G_2)$ is bounded and bounded away from zero. Hence, there exists $F_r \in C_b(\mathbb{A}_r)$ such that $\det(G_1^*(z) G_2(z)) F_r(z) = 1$, for $z \in \mathbb{A}_r$. That is, for ρ large enough $\det(G_1^* G_2) \in \text{inv}(C_b(\mathbb{A}_\rho))$. Moreover, if $W(\det(G_1^* G_2)) = 0$, then, arguing as when we verified assumption (A3),

$$\lim_{r \rightarrow 1} w(\det(G_1^*(z) G_2(z))|_{\mathbb{T}_r}) = 0.$$

Due to the local constancy of the winding number, this means that $w(\det(G_1^* G_2)|_{\mathbb{T}_r}) = 0$ for all r close enough to 1. That is, $\det(G_1^* G_2) \in \text{inv } C_b(\mathbb{A}_\rho)$ and $w(\det(G_1^* G_2)|_{\mathbb{T}_r}) = 0$, $r \in (\rho, 1)$, for all ρ close enough to 1. So by Theorem 4.2, d_ν and \tilde{d}_ν coincide in this case.

Next, let us assume that $\tilde{d}_\nu(P_1, P_2) < 1$. Then $\tilde{d}_\nu^\rho(P_1, P_2) < 1$ for all ρ sufficiently close to 1, which means that $\det(G_1^* G_2) \in \text{inv } C_b(\mathbb{A}_\rho)$ and that $w(\det(G_1^* G_2)|_{\mathbb{T}_r}) = 0$ for $r \in (\rho, 1)$. Therefore, there exists $(F_r) \in \mathcal{A}$ such that $\det(G_1^* G_2)|_{\mathbb{A}_r} \cdot F_r = 1$ pointwise for $r \in (\rho, 1)$.

Since π_r is a $*$ -homomorphism, this implies that $\det(G_1^*G_2)$ is invertible as an element of $\varinjlim C_b(\mathbb{A}_r)$. By definition,

$$W(\det(G_1^*G_2)) = \lim_{r \rightarrow 1} w(\det(G_1^*G_2)|_{\mathbb{T}_r}),$$

and so the assumption that $w(\det(G_1^*G_2)|_{\mathbb{T}_r}) = 0$, $r \in (\rho, 1)$, for all ρ sufficiently close to 1 implies that $W(\det(G_1^*G_2)) = 0$. That is, in view of Theorem 4.2, d_ν and \tilde{d}_ν coincides also in this case, which completes the proof. \square

6. A COMPUTATIONAL EXAMPLE

As an illustration of the computability of the proposed ν -metric, we give an example where we calculate explicitly the ν -metric when there is uncertainty in the location of the zero of the (nonrational) transfer function.

In [6], it was shown that

$$\begin{aligned} d_\nu \left(e^{-sT} \frac{s}{s-a_1}, e^{-sT} \frac{s}{s-a_2} \right) &= \frac{|a_1 - a_2|}{\sqrt{2}(a_1 + a_2)} \text{ when } |a_1 - a_2| \text{ is small enough, while} \\ d_\nu \left(e^{-sT_1} \frac{s}{s-a}, e^{-sT_2} \frac{s}{s-a} \right) &= 1 \text{ whenever } T_1 \neq T_2. \end{aligned}$$

Continuing this theme, we will now calculate

$$d_\nu \left(e^{-sT} \frac{s-a_1}{s-b}, e^{-sT} \frac{s-a_2}{s-b} \right),$$

hence quantifying the effect of uncertainty in the *zero* location, and complementing the previous two computations done in [6], where the effects of uncertainty in the *pole* location, and uncertainty in the *delay* were found out.

Consider the transfer function P given by

$$P(s) := e^{-sT} \frac{s-a}{s-b}, \tag{6.1}$$

where $T, b > 0$, $a \in \mathbb{R}$, and $a \neq b$. Then $P \in \mathbb{F}(H^\infty(\mathbb{C}_{>0}))$, where $H^\infty(\mathbb{C}_{>0})$ denotes the set of bounded and holomorphic functions defined in the open right half plane

$$\mathbb{C}_{>0} := \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}.$$

Using the conformal map $\varphi : \mathbb{D} \rightarrow \mathbb{C}_{>0}$,

$$\varphi(z) = \frac{1+z}{1-z},$$

we can transplant the plant to \mathbb{D} . In this manner, we can also talk about a ν -metric on $\mathbb{S}(H^\infty(\mathbb{C}_{>0}), p, m)$.

We will calculate the distance between a pair of plants arising from (6.1), when there is uncertainty in the parameter a , the zero of the transfer function. A normalized (left and right) coprime factorization of P is given by $P = N/D$, where

$$N(s) = \frac{(s-b)e^{-sT}}{\sqrt{2}s + \sqrt{a^2 + b^2}}, \quad D(s) = \frac{s-a}{\sqrt{2}s + \sqrt{a^2 + b^2}}.$$

This factorization was found using the algorithm given in [3, Example 4.1]. Set $s := \varphi(z)$ for $z \in \mathbb{D}$, and consider the two plants

$$P_1 := e^{-sT} \frac{s - a_1}{s - b} \quad \text{and} \quad P_2 := e^{-sT} \frac{s - a_2}{s - b}, \quad (6.2)$$

where $T, b > 0$ and $a_1, a_2 \in \mathbb{R} \setminus \{b\}$. Define f by

$$f(s) := G_1^* G_2 = \overline{N}_1 N_2 + \overline{D}_1 D_2 = \frac{(\overline{s} - b)(s - b)e^{-2\operatorname{Re}(s)T} + (\overline{s} - a_1)(s - a_2)}{(\sqrt{2}\overline{s} + \sqrt{a_1^2 + b^2})(\sqrt{2}s + \sqrt{a_2^2 + b^2})}. \quad (6.3)$$

Note that the map $z \mapsto |f(\varphi(z))|$ is bounded on \mathbb{D} . We shall show that the real part of this map is nonnegative and bounded away from zero for all $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 0$, provided that $|a_1 - a_2|$ is small enough. The proof of this fact is analogous to the proof of [6, Lemma 4.1].

Lemma 6.1. *Let $T, b > 0$ and $a_1, a_2 \in \mathbb{R} \setminus \{b\}$. Set $\mathbb{C}_{>0} = \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$. Let $f(s)$ be defined as in (6.3). Then there exist δ_0 and $m > 0$ such that for all $\delta \in [0, \delta_0)$, $s \in \mathbb{C}_{>0}$, there holds that: if $|a_1 - a_2| < \delta$, then $\operatorname{Re}(f(s)) > m > 0$.*

Proof. Without loss of generality, we may assume that $a_1 < a_2$. Let a and δ be such that $a = a_1$ and $a + \delta = a_2$ respectively, and choose $\varepsilon > 0$ such that $\frac{\varepsilon^2}{2} + \frac{3\varepsilon}{2\sqrt{2}} < \frac{1}{4}$. Note that

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in \mathbb{C}_{>0}}} \frac{s - a}{\sqrt{2}s + \sqrt{a^2 + b^2}} = \frac{1}{\sqrt{2}}.$$

Therefore we can choose $R > 0$ such that

$$\left| \frac{s - a}{\sqrt{2}s + \sqrt{a^2 + b^2}} - \frac{1}{\sqrt{2}} \right| < \frac{\varepsilon}{2}, \quad (6.4)$$

for all $|s| > R$. We have

$$\begin{aligned} I &:= \left| \frac{s - a}{\sqrt{2}s + \sqrt{a^2 + b^2}} - \frac{s - (a + \delta)}{\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}} \right| \\ &= \left| \frac{\delta(\sqrt{2}s + \sqrt{a^2 + b^2}) + (s - a)(\sqrt{(a + \delta)^2 + b^2} - \sqrt{a^2 + b^2})}{(\sqrt{2}s + \sqrt{a^2 + b^2})(\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2})} \right| \end{aligned}$$

Let us assume that $\delta < |a|$. Then we have

$$\begin{aligned} I &\leq \delta \frac{|\sqrt{2}s + \sqrt{a^2 + b^2}| + 3|s - a|}{|\sqrt{2}s + \sqrt{a^2 + b^2}| |\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}|} \\ &\leq \delta \left(\frac{1}{|\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}|} + \left| \frac{s - a}{\sqrt{2}s + \sqrt{a^2 + b^2}} \right| \cdot \frac{3}{|\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}|} \right) \\ &\leq \delta \left(\frac{1}{\sqrt{2}R} + \left(\frac{1}{\sqrt{2}} + \frac{\varepsilon}{2} \right) \frac{3}{\sqrt{2}R} \right), \end{aligned}$$

since $\operatorname{Re}(s) > 0$. Therefore, if we choose δ_0 so that for all $0 \leq \delta < \delta_0$,

$$\delta < \frac{\sqrt{2}R}{1 + \frac{3}{\sqrt{2}} + \frac{3\varepsilon}{2}} \cdot \frac{\varepsilon}{2},$$

then for all such δ and for $|s| > R$, we have that

$$\left| \frac{s - (a + \delta)}{\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}} - \frac{1}{\sqrt{2}} \right| < \varepsilon. \quad (6.5)$$

As a consequence of (6.4) and (6.5), and using

$$xy - \frac{1}{2} = \left(x - \frac{1}{\sqrt{2}}\right) \left(y - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \left(x - \frac{1}{\sqrt{2}} + y - \frac{1}{\sqrt{2}}\right),$$

we obtain

$$\left| \frac{s - a}{\sqrt{2}s + \sqrt{a^2 + b^2}} \cdot \frac{s - (a + \delta)}{\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}} - \frac{1}{2} \right| \leq \frac{\varepsilon^2}{2} + \frac{3\varepsilon}{2\sqrt{2}}.$$

Therefore,

$$\frac{1}{2} - \operatorname{Re} \left(\frac{\bar{s} - a}{\sqrt{2}\bar{s} + \sqrt{a^2 + b^2}} \cdot \frac{s - (a + \delta)}{\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}} \right) < \frac{1}{4},$$

and

$$\operatorname{Re} \left(\frac{\bar{s} - a}{\sqrt{2}\bar{s} + \sqrt{a^2 + b^2}} \cdot \frac{s - (a + \delta)}{\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2}} \right) > \frac{1}{4}.$$

Also, for $s \in \mathbb{C}_{>0}$,

$$\operatorname{Re} \left(\frac{|s - b|^2 e^{-2\operatorname{Re}(s)T}}{(\sqrt{2}\bar{s} + \sqrt{a^2 + b^2})(\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2})} \right) > 0.$$

Combined, this means that, for $|s| > R$ and $0 \leq \delta < \delta_0$, $\operatorname{Re}(f(s)) > \frac{1}{4}$.

To treat the case when $|s| < R$, set $K = \{s \in \mathbb{C}_{\geq 0} : |s| \leq R\}$. Define $F : K \times [0, 1] \rightarrow \mathbb{R}$ by

$$F(s, \delta) = \operatorname{Re} \left(\frac{(\bar{s} - b)(s - b)e^{-2\operatorname{Re}(s)T} + (\bar{s} - a)(s - (a + \delta))}{(\sqrt{2}\bar{s} + \sqrt{a^2 + b^2})(\sqrt{2}s + \sqrt{(a + \delta)^2 + b^2})} \right).$$

Then

$$F(s, 0) = \operatorname{Re} \left(\frac{|s - b|e^{-2\operatorname{Re}(s)T} + |s - a|^2}{|\sqrt{2}s + \sqrt{a^2 + b^2}|} \right) \geq 0$$

and let $2m := \min_{s \in K} F(s, 0)$. Clearly $m \geq 0$, and in fact, since $a \neq b$, $|s - a|^2$ and $|s - b|^2$ cannot be zero simultaneously so $m > 0$. Now, since F is continuous on the compact set $K \times [0, 1]$, F is uniformly continuous there. This means that we may, if necessary, redefine our choice of δ_0 so that if $0 \leq \delta < \delta_0$, then $|F(s, \delta) - F(s, 0)| < m$ for all $s \in K$. That is, $F(s, \delta) = \operatorname{Re}(f(s)) > m$ for all $\delta \in [0, \delta_0)$, $s \in K$. Combining these observations, we see that $\operatorname{Re}(f(s)) > \min\{m, 1/4\}$. This completes the proof. \square

As a consequence of this result, $G_1^* G_2$ is invertible (as an element of $\varinjlim C_b(\mathbb{A}_r)$) and its index is $W(G_1^* G_2) = 0$. Thus the distance between the two plants P_1, P_2 in (6.2) is given by

$$\begin{aligned} \|\tilde{G}_2 G_1\|_\infty &= \sup_{s=i\omega, \omega \in \mathbb{R}} \left| \frac{(a_1 - a_2)(s - b)e^{-sT}}{(\sqrt{2}s + \sqrt{a_1^2 + b^2})(\sqrt{2}s + \sqrt{a_2^2 + b^2})} \right| \\ &= \frac{|a_1 - a_2|}{2} \sup_{\omega \in \mathbb{R}} \frac{\sqrt{\omega^2 + b^2}}{\sqrt{\omega^2 + \frac{a_1^2 + b^2}{2}} \sqrt{\omega^2 + \frac{a_2^2 + b^2}{2}}}. \end{aligned}$$

We will now determine the supremum in the last expression in the following two mutually exhaustive cases:

1° Suppose that $(a_1^2 - b^2)(a_2^2 - b^2) \geq 4b^4$. Then we have

$$\begin{aligned} \sup_{\omega \in \mathbb{R}} \frac{\sqrt{\omega^2 + b^2}}{\sqrt{\omega^2 + \frac{a_1^2 + b^2}{2}} \sqrt{\omega^2 + \frac{a_2^2 + b^2}{2}}} &= \sup_{\omega \in \mathbb{R}} \frac{1}{\sqrt{(\omega^2 + b^2) + \frac{a_1^2 + a_2^2 - 2b^2}{2} + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4(\omega^2 + b^2)}}} \\ &= \frac{1}{\sqrt{\inf_{\omega \in \mathbb{R}} \left(\omega^2 + b^2 + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4(\omega^2 + b^2)} \right) + \frac{a_1^2 + a_2^2 - 2b^2}{2}}}. \end{aligned}$$

By the arithmetic mean-geometric mean inequality,

$$\omega^2 + b^2 + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4(\omega^2 + b^2)} \geq \sqrt{(a_1^2 - b^2)(a_2^2 - b^2)},$$

with equality if and only if

$$\omega^2 = \sqrt{\frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4}} - b^2 \geq 0,$$

thanks to our assumption that $(a_1^2 - b^2)(a_2^2 - b^2) \geq 4b^4$. Thus

$$\sup_{\omega \in \mathbb{R}} \frac{\sqrt{\omega^2 + b^2}}{\sqrt{\omega^2 + \frac{a_1^2 + b^2}{2}} \sqrt{\omega^2 + \frac{a_2^2 + b^2}{2}}} = \frac{\sqrt{2}}{\sqrt{a_1^2 - b^2} + \sqrt{a_2^2 - b^2}}.$$

Note that in the above, we have used the fact that $a_1^2 > b^2$ and $a_2^2 > b^2$ which follows from the condition $(a_1^2 - b^2)(a_2^2 - b^2) \geq 4b^4$ (≥ 0): indeed, if we have (the only other case) $b^2 \geq a_1^2$ and $b^2 \geq a_2^2$, then we arrive at the contradiction that $b^4 = b^2 \cdot b^2 > (b^2 - a_1^2)(b^2 - a_2^2) \geq 4b^4$.

2° Now let us consider the other possibility, namely that $(a_1^2 - b^2)(a_2^2 - b^2) < 4b^4$. Then for $\omega \in \mathbb{R}$, we have $4b^2(\omega^2 + b^2) \geq 4b^4 > (a_1^2 - b^2)(a_2^2 - b^2)$ and so

$$1 > \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4b^2(\omega^2 + b^2)}.$$

From this we obtain upon multiplying both sides by ω^2 (≥ 0) that

$$\omega^2 \geq \frac{(a_1^2 - b^2)(a_2^2 - b^2)\omega^2}{4b^2(\omega^2 + b^2)} = \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4} \left(\frac{1}{b^2} - \frac{1}{\omega^2 + b^2} \right).$$

By rearranging and adding b^2 on both sides, we have

$$\omega^2 + b^2 + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4(\omega^2 + b^2)} \geq b^2 + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4b^2},$$

and so

$$\begin{aligned} \frac{1}{\sqrt{(\omega^2 + b^2) + \frac{a_1^2 + a_2^2 - 2b^2}{2} + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4(\omega^2 + b^2)}}} &\leq \frac{1}{\sqrt{b^2 + \frac{(a_1^2 - b^2)(a_2^2 - b^2)}{4b^2} + \frac{a_1^2 + a_2^2 - 2b^2}{2}}} \\ &= \frac{2b}{\sqrt{a_1^2 + b^2} \sqrt{a_2^2 + b^2}} \end{aligned}$$

and there is equality if $\omega = 0$. Consequently,

$$\sup_{\omega \in \mathbb{R}} \frac{\sqrt{\omega^2 + b^2}}{\sqrt{\omega^2 + \frac{a_1^2 + b^2}{2}} \sqrt{\omega^2 + \frac{a_2^2 + b^2}{2}}} = \frac{2b}{\sqrt{a_1^2 + b^2} \sqrt{a_2^2 + b^2}}.$$

Summarizing the two cases, we have

$$d_\nu \left(e^{-sT} \frac{s - a_1}{s - b}, e^{-sT} \frac{s - a_2}{s - b} \right) = \begin{cases} \frac{|a_1 - a_2|}{\sqrt{2}(\sqrt{a_1^2 - b^2} + \sqrt{a_2^2 - b^2})} & \text{if } (a_1^2 - b^2)(a_2^2 - b^2) \geq 4b^4 \\ \frac{b|a_1 - a_2|}{\sqrt{a_1^2 + b^2} \sqrt{a_2^2 + b^2}} & \text{if } (a_1^2 - b^2)(a_2^2 - b^2) < 4b^4 \end{cases}$$

when $|a_1 - a_2|$ is small enough.

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